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Expressing the Algebraic of Triangular Number in Sequence and Series to the Sum of Arithmetic Progression

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Abstract

The formulae for triangular number as an alternative for the sum of arithmetic progression was further demonstrated through routine algebraic procedures which resulted in the same formulae with mathematical reasonings and inductions for its nth term that was later validated using the R version 3.6.1

Key words: Triangular number, the sum of arithmetic progression, mathematical induction, the sum of positive integers.

1. Introduction

Recent contemporary studies on the theories of numbers might have paved the way for mathematical discoveries and fascinating properties of triangular numbers for more detailed explorations in science, engineering and technology. Number theory as an integral part of mathematics with a contesting problem in series and sequences which serve as a source of attraction to many mathematicians paving ways for the study of number sequence, for example, cardinal numbers as part of figurative numbers in theories of number can, of course, be studied as a mathematical topic (Panda & Ray, 2011; Kleiner, 2012).

Arithmetic and geometry seem to be antithetical at first sight, one dealing with the discrete and the other with the continuous. The relations between the two are, however deep, though often hidden. The tensions between number and geometry and between the related analytic and synthetic approaches to mathematics have been very beneficial for the development of the subject (Olds, et al., 2000, Stillwell, 2002 and Mazur, 2003).

Although the connection between arithmetic and geometry is fundamental, it has not always been amicable. The early Greek harmony between number and shape, given expression in, among other things, the arithmetic development of the Pythagorean theory of similarity, was shattered by the Greek crisis of incommensurability, that is, by the proof of the existence of incommensurable magnitudes (Kline, 1972). However, an example of this cooperative relationship was the introduction by the Pythagoreans of the polygonal numbers which gave birth to triangular numbers as our focusing point in this study.

Beldon and Gardiner (2002) compared triangular numbers with perfect squares and established that the sum of any two consecutive triangular numbers is always a perfect square. This fact was known to the ancient Greeks and attributed the result to Theon of Smyrna (Heath, 1981) with the name triangular numbers stemming from the Greek interest in figurative numbers whereas interest in square triangular numbers by Burn (1991) metamorphosed into continued fractions.

Pythagoreans study on polygonal numbers depicted triangular numbers as being represented by a triangular array of dots and sum of positive integers (Tattersall, 1999) but each primitive Pythagorean triple of positive integers leads with the aid of an algebraic identity to a family of triples of integers. However, each triple in this family provides three triangular numbers, one for each component, such that the sum of two of them is equal to the third one (Haggard, 1997). This study tends to use the dynamism and flexibility nature of triangular numbers to investigate its link with the sum of arithmetic progression inasmuch triangular numbers are sequent partial sums of positive integers $\mathbb{Z}^+$.

2. TRIANGULAR NUMBERS TO SUM OF ARITHMETIC PROGRESSION

Let the partial sums of positive integers be equivalent to the triangular numbers proposed in Afolabi and Oluwagunwa (2021) & Oluwagunwa and Afolabi (2018) that the partial sum represented by (S_n^*) for the sequence of positive integers $(\mathbb{Z}^+)$ is both equal and the same in algebraic formulae and numerical values for triangular numbers (t_n).

$$S_n^* = t_n = \sum_{k=1}^{n} k$$

$$S_1^* = t_1 = \sum_{k=1}^{1} k = 1$$

$$S_2^* = t_2 = \sum_{k=1}^{2} k = 1 + 2 = 3$$

$$S_3^* = t_3 = \sum_{k=1}^{3} k = 1 + 2 + 3 = 6$$

$$\vdots$$

$$S_n^* = t_n = \sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots + n$$

Suppose an arithmetic progression is strictly an increasing sequence of the form $a_1, a_2, a_3, \cdots, a_n, a_{n+1}, \cdots$ with the difference of any two successive members being a constant (d), then

$$a_2 - a_1 = a_3 - a_2 = \cdots = a_n - a_{n-1} = a_{n+1} - a_n = \cdots = d$$

It, therefore, implies that the common difference (d) between any successive terms of the arithmetic progression can be written in general form or term (T_n) as

$$T_n = a_1 + (n - 1)d$$

with the sum (S_n) of the first n terms of this sequence

$$S_n = a_1 + a_2 + \cdots + a_n = \frac{n(a_1 + a_n)}{2}$$

amounting to the submission of Oluwagunwa and Afolabi (2018) on mathematical discoveries and fascinating properties of triangular numbers in partial sums of positive integers and sum of arithmetic progression.

Let

$$S_n = a_1 + a_2 + \cdots + a_n$$

or

$$S_n = a_n + a_{n-1} + \cdots + a_1$$

here

$$a_1 = a$$
$$a_2 = a + d$$
$$a_3 = a + 2d$$
$$a_4 = a + 3d$$
$$a_5 = a + 4d$$
$$\vdots$$
$$a_n = a + (n - 1)d$$
$$S_n = a + (a + d) + (a + 2d) + \cdots + [a + (n - 1)d]$$

or

$$S_n = [a + (n - 1)d] + [a + (n - 2)d] + [a + (n - 3)d] + \cdots + a$$

The Sum of this arithmetic progression could either be ascending or descending order. Therefore,

$$S_n + S_n = \{a + [a + (n-1)d]\} + \{(a+d) + [a + (n-2)d]\} + \cdots + \{[a + (n-1)d] + a\}$$
$$2S_n = [2a + (n-1)d] + [2a + (n-1)d] + \cdots + [2a + (n-1)d] n\ times$$
$$2S_n = n[2a + (n-1)d]$$
$$S_n = \frac{n}{2}[2a + (n-1)d] \tag{1}$$

Suppose the sequence $a_1, a_2, a_3, \cdots, a_n$ is a series of positive integers ($\mathbb{Z}^+$)

$$1 + 2 + 3 + \cdots + n$$

then,

$$S_1^* = a_1 = a = 1 = t_1$$
$$S_2^* = a_1 + a_2 = a + (a+d) = 1 + 2 = t_2 \ (since\ a_2 - a_1 = d)$$
$$S_3^* = a_1 + a_2 + a_3 = a + (a+d) + (a+2d) = 1 + 2 + 3 = t_3$$
$$\vdots$$
$$S_n^* = a_1 + a_2 + a_3 + \cdots + a_n = a + (a+d) + (a+2d) + \cdots + [a + (n-1)d] = 1 + 2 + 3 + \cdots + n = t_n$$

Comparing S_n^* to t_n, we have

$$S_1^* = t_1 = a_1 = 1 = a$$
$$S_2^* = t_2 = a_1 + a_2 = 1 + 2 = 3 = a + (a+d) = 2a + d$$
$$S_3^* = t_3 = a_1 + a_2 + a_3 = 1 + 2 + 3 = 6 = a + (a+d) + (a+2d) = 3a + 3d$$
$$S_4^* = t_4 = a_1 + a_2 + a_3 + a_4 = 1 + 2 + 3 + 4 = 10 = a + (a+d) + (a+2d) + (a+3d) = 4a + 6d$$
$$S_5^* = t_5 = a_1 + a_2 + a_3 + a_4 + a_5 = 1 + 2 + 3 + 4 + 5 = 15 = a + (a+d) + (a+2d) + (a+3d) + (a+4d) = 5a + 10d$$

by mathematical induction, we apply $\binom{n}{1}$ and $\binom{n}{2}$ for coefficients of a and d respectively,

then

$$S_1^* = t_1 = \binom{1}{1} a = a$$

and coefficient of d will not exist, since $\binom{n}{k}$ is only for $k \leq n$

$$S_2^* = t_2 = \binom{2}{1} a + \binom{2}{2} d = 2a + d$$
$$S_3^* = t_3 = \binom{3}{1} a + \binom{3}{2} d = 3a + 3d$$
$$S_4^* = t_4 = \binom{4}{1} a + \binom{4}{2} d = 4a + 6d$$
$$S_5^* = t_5 = \binom{5}{1} a + \binom{5}{2} d = 5a + 10d$$
$$\vdots$$
$$S_n^* = t_n = \binom{n}{1} a + \binom{n}{2} d = \frac{n!}{(n-1)!\,1!} a + \frac{n!}{(n-2)!\,2!} d = na + \frac{n(n-1)}{2} d$$

Hence, the sum of this arithmetic progression (S_n) is deduced from the algebraic formulae of a triangular number (t_n) as

$$S_n = t_n = na + n(n-1)\frac{d}{2} \tag{2}$$

For $a = d = 1$

$$S_1^* = t_1 = a_1 = 1 = a \ (a \Rightarrow first\ term)$$
$$S_2^* = t_2 = a_1 + a_2 = 1 + 2 = 3 = 2 + 1 = 2a + 1 \ (for\ d = 1)$$
$$S_3^* = t_3 = a_1 + a_2 + a_3 = 1 + 2 + 3 = 6 = 3 + 3 = 3a + 3(1)$$
$$S_4^* = t_4 = a_1 + a_2 + a_3 + a_4 = 1 + 2 + 3 + 4 = 10 = 4 + 6 = 4a + 6(1)$$
$$S_5^* = t_5 = a_1 + a_2 + a_3 + a_4 + a_5 = 1 + 2 + 3 + 4 + 5 = 15 = 5 + 10 = 5a + 10(1)$$

by mathematical reasoning, coefficients of a follow natural numbers (n) from S_1^* to S_5^* while coefficients of common difference ($d = 1$) follow triangular numbers in succession from S_2^* to S_5^* indicating a lagged by 1.

Since $t_n = \frac{n(n+1)}{2}$ then $t_{n-1} = \frac{(n-1)(n+1-1)}{2} = \frac{n(n-1)}{2}$

hence,

$$S_n^* = na + t_{n-1}(1) = na + \frac{n(n-1)}{2}(d)$$
$$S_n^* = na + \frac{n(n-1)}{2}(d) = na + n(n-1)\frac{d}{2} = S_n = t_n \tag{3}$$

Therefore,

$$t_n = an + \frac{n(n-1)d}{2}$$

if $a = d = 1$, then $t_n = n + t_{n-1}$

$$t_n = n + \frac{n(n-1)}{2} = \frac{2n + n^2 - n}{2}$$

$$t_n = \frac{n(n+1)}{2}$$

3. PROBLEM DEMONSTRATIONS

Suppose $a_1, a_2, a_3, \cdots, a_n$ is an arithmetic progression with common difference d, we are expected to find in terms of d and a_1 & a_n the explicit value of the sum

$$S_n^* = \frac{1}{a_1 a_2} + \frac{1}{a_2 a_3} + \cdots + \frac{1}{a_{n-2} a_{n-1}} + \frac{1}{a_{n-1} a_n}$$

Note that

$$\frac{1}{a(a+d)} = \frac{1}{d}\left(\frac{1}{a} - \frac{1}{a+d}\right)$$

then S_n^* can be written in a telescoping sum for which everything cancels except the first and the last terms

$$S_n^* = \frac{1}{d}\left[\left(\frac{1}{a_1} - \frac{1}{a_2}\right) + \left(\frac{1}{a_2} - \frac{1}{a_3}\right) + \cdots + \left(\frac{1}{a_{n-2}} - \frac{1}{a_{n-1}}\right) + \left(\frac{1}{a_{n-1}} - \frac{1}{a_n}\right)\right] = \frac{1}{d}\left(\frac{1}{a_1} - \frac{1}{a_n}\right)$$

and in particular, if $a_n = n$ then

$$S_n^* = \frac{1}{(1)(2)} + \frac{1}{(2)(3)} + \cdots + \frac{1}{(n-2)(n-1)} + \frac{1}{(n-1)(n)} = \frac{n-1}{n}$$

Given that there are no arithmetic progressions of positive integers whose terms are all perfect squares. By contradiction, $\exists$ positive integers

$$a_1 < a_2 < \cdots < a_{n-1} < a_n < a_{n+1} < \cdots$$

such that

$$a_1^2 < a_2^2 < \cdots < a_{n-1}^2 < a_n^2 < a_{n+1}^2 < \cdots$$

is said to be an arithmetic progression with common difference d

$$d = a_2^2 - a_1^2 = a_3^2 - a_2^2 = \cdots = a_n^2 - a_{n-1}^2 = a_{n+1}^2 - a_n^2 = \cdots$$

it follows that

$$(a_n - a_{n-1})(a_n + a_{n-1}) = (a_{n+1} - a_n)(a_{n+1} + a_n), \qquad n = 2,3,4, \ldots$$

since $a_{n-1} < a_n < a_{n+1} \Rightarrow a_{n+1} > a_n > a_{n-1}$ so the above equality gives

$$a_2 - a_1 > a_3 - a_2 > a_4 - a_3 > \cdots > a_n - a_{n-1} > \cdots > 0$$

which is clearly impossible as this condition is shown by contradiction.

4. PRACTICAL DEMONSTRATIONS WITH THE R SOFTWARE

```
### Function to call out TRIANGULAR NUMBER through PARTIAL SUM OF INTEGERS ###
    arithsum = function(a,d,n){
          y=0
          for (i in 1:n) y=(i*a)+i*(i-1)*d/2
          return(y)
    }
    ### TRIANGULAR NUMBER from FIRST to TWENTIETH position in its series ###
    arithsum(1,1,1)
    arithsum(1,1,2)
    arithsum(1,1,3)
    arithsum(1,1,4)
    arithsum(1,1,5)
    arithsum(1,1,6)
    arithsum(1,1,7)
    arithsum(1,1,8)
    arithsum(1,1,9)
    arithsum(1,1,10)
```

```
arithsum(1,1,11)
arithsum(1,1,12)
arithsum(1,1,13)
arithsum(1,1,14)
arithsum(1,1,15)
arithsum(1,1,16)
arithsum(1,1,17)
arithsum(1,1,18)
arithsum(1,1,19)
arithsum(1,1,20)
### Few Results ###
> arithsum(1,1,1)
[1] 1
> arithsum(1,1,2)
[1] 3
> arithsum(1,1,3)
[1] 6
> arithsum(1,1,11)
[1] 66
> arithsum(1,1,16)
[1] 136
> arithsum(1,1,20)
[1] 210

### Function to call out SUM OF ARITHMETIC PROGRESSION for TRIANGULAR NUMBER ###
arithsumtriangle = function(a,d,n){
        y=0
        for (i in 1:n) y=y+(i*a)+i*(i-1)*d/2
        return(y)
}
### SUM of A. P. for TRIANGULAR NUMBER from FIRST to TWENTIETH position ###
arithsumtriangle(1,1,1)
arithsumtriangle(1,1,2)
arithsumtriangle(1,1,3)
arithsumtriangle(1,1,4)
arithsumtriangle(1,1,5)
arithsumtriangle(1,1,6)
arithsumtriangle(1,1,7)
arithsumtriangle(1,1,8)
arithsumtriangle(1,1,9)
arithsumtriangle(1,1,10)
arithsumtriangle(1,1,11)
arithsumtriangle(1,1,12)
arithsumtriangle(1,1,13)
arithsumtriangle(1,1,14)
arithsumtriangle(1,1,15)
arithsumtriangle(1,1,16)
arithsumtriangle(1,1,17)
arithsumtriangle(1,1,18)
arithsumtriangle(1,1,19)
arithsumtriangle(1,1,20)
### Few Results ###
> arithsumtriangle(1,1,1)
[1] 1
```

```
> arithsumtriangle(1,1,2)
[1] 4
> arithsumtriangle(1,1,3)
[1] 10
> arithsumtriangle(1,1,8)
[1] 120
> arithsumtriangle(1,1,12)
[1] 364
> arithsumtriangle(1,1,15)
[1] 680
> arithsumtriangle(1,1,19)
[1] 1330
> arithsumtriangle(1,1,20)
[1] 1540
```

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